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Lie symmetries and exact solutions of first-order difference schemes

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Abstract

We show that any first-order ordinary differential equation with a known Lie point symmetry group can be discretized into a difference scheme with the same symmetry group. In general, the lattices are not regular ones, but must be adapted to the symmetries considered. The invariant difference schemes can be so chosen that their solutions coincide exactly with those of the original differential equation.

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1. Introduction

The purpose of this paper is to analyse Lie point symmetries and to obtain the exact solutions of first-order difference schemes. These are two-point schemes of the form

$$E_a(x, y, x_+, y_+) = 0 \qquad a = 1, 2 \tag{1.1}$$

$$\left|\frac{\partial(E_1, E_2)}{\partial(x_+, y_+)}\right| \neq 0 \tag{1.2}$$

where we use the notation $x \equiv x_n, x_+ \equiv x_{n+1}, y \equiv y_n, y_+ \equiv y_{n+1}$ for the independent and dependent variables, evaluated at two different points. The two equations (1.1) define a difference equation, as well as a lattice. The general solution of the scheme depends on two constants and has the form

$$y = y(n, C_1, C_2) \qquad x = x(n, C_1, C_2). \tag{1.3}$$

Equation (1.3) can also be rewritten as

$$y = y(x, C_1, C_2) \qquad x = x(n, C_1, C_2). \tag{1.4}$$

The formula for y can be interpreted as interpolating between the points of the lattice and determining the solution y for all values of x.

A standard equally spaced lattice is given by a specific choice of one of the two equations (1.1), namely, $E_2 = x_+ - x - h = 0$, where h is a constant (the lattice spacing) and its solution is $x_n = nh + x_0$ (where x_0 is one of the two integration constants of the scheme).

Instead of the variables x, y, x_+, y_+, we can use

$$x \qquad y \qquad h = x_+ - x \qquad y_x = \frac{y_+ - y}{x_+ - x}. \tag{1.5}$$

The continuous limit $h \to 0$ of the scheme (1.1) is then obvious, namely, one of the equations should turn into a first-order ordinary differential equation (ODE), the other into an identity (like $0 = 0$). We shall write the ODE as

$$E(x, y, y') = y' - F(x, y) = 0. \tag{1.6}$$

The symmetries and solutions of the difference scheme (1.1) that we shall obtain below will in the limit $h \to 0$ turn into Lie point symmetries and solutions of the ODE (1.6). To establish the connection, let us recall some well-known results on symmetries of first-order ODEs [1].

The Lie point symmetry group of the ODE (1.6) is always infinite-dimensional [1]. Its Lie algebra, the 'symmetry algebra', is realized by vector fields of the form

$$X = \xi(x, y)\partial_x + \phi(x, y)\partial_y \tag{1.7}$$

satisfying

$$\operatorname{pr} X(E)|_{E=0} = 0. \tag{1.8}$$

In (1.8), $\operatorname{pr} X$ is the first prolongation of X, i.e. [1]

$$\begin{aligned} \operatorname{pr} X &= \xi(x, y)\partial_x + \phi(x, y)\partial_y + \phi^x(x, y, y')\partial_{y'} \\ \phi^x &= \phi_x + (\phi_y - \xi_x)y' - \xi_y(y')^2 \end{aligned} \tag{1.9}$$

where the subscripts are partial derivatives. Equation (1.8) amounts to a single first-order linear partial differential equation for the two functions ξ and ϕ and as such it has infinitely many solutions. This is the reason why the Lie point symmetry of equation (1.6) is infinite-dimensional [1]. In general, it may be difficult or impossible to find any explicit analytical solution, as difficult as finding an integrating multiplier. However, if we find at least one particular explicit solution of equation (1.8), we can obtain a one-dimensional subalgebra of the symmetry algebra of equation (1.6). This is sufficient to integrate equation (1.6) in quadratures.

All elementary methods of solving first-order ODEs amount to special cases of the above procedure.

A different application of the vector field X and its prolongation (1.9) is to construct first-order ODEs that are invariant under a given Lie group of local point transformations, namely, those generated by the vector field X. In this case, the functions $\xi(x, y)$, $\phi(x, y)$ and hence also $\phi^x(x, y, y')$, are known. The invariant equation is obtained by solving the first-order partial differential equation:

$$[\xi(x, y)\partial_x + \phi(x, y)\partial_y + \phi^x(x, y, y')\partial_{y'}]E(x, y, y') = 0 \tag{1.10}$$

for the function $E(x, y, y')$. Solving by the method of characteristics, we obtain two elementary invariants:

$$I_1 = I_1(x, y) \qquad I_2 = I_2(x, y, y') \qquad \frac{\partial I_2}{\partial y'} \neq 0. \tag{1.11}$$

An invariant equation is given by any relation between I_1 and I_2, i.e.,

$$E(I_1, I_2) = 0 \qquad \frac{\partial E}{\partial I_2} \neq 0. \tag{1.12}$$

If we are given a two-dimensional Lie algebra of vector fields $\{X_1, X_2\}$ and we require invariance under the two-dimensional Lie group they generate, then two different possibilities can occur. The first is that the two equations (1.10) (for X_1 and X_2, respectively) have a common solution

$$I = I(x, y, y') \qquad \frac{\partial I}{\partial y'} \neq 0. \tag{1.13}$$

The invariant ODE then is

$$F(I) = 0 \tag{1.14}$$

where F is arbitrary. We say that equation (1.14) is 'strongly invariant' with respect to the group generated by $\{X_1, X_2\}$. If no such invariant $I(x, y, y')$ exists, then we look for an invariant manifold and a 'weakly invariant' equation. This is obtained from the condition that the two equations $\mathrm{pr}\, X_1(E) = 0$, $\mathrm{pr}\, X_2(E) = 0$ should be equivalent. This is a condition on the matrix of coefficients, i.e.,

$$\mathrm{rank} \begin{pmatrix} \xi_1 & \phi_1 & \phi_1^x \\ \xi_2 & \phi_2 & \phi_2^x \end{pmatrix} = 1. \tag{1.15}$$

This condition, together with, say, $\mathrm{pr}\, X_1(E) = 0$, provides the weakly invariant equation.

In section 2, we will adapt the above results to the case of the difference system (1.1) and in sections 3–9 consider many examples. The examples will be difference analogues of ODEs with known symmetry groups (linear equations, separable equations, etc).

In each case, we first present a one- or two-dimensional symmetry algebra of the ODE and use it to solve the equation. We have not found these symmetry algebras in the literature, but they are implicit in standard integration procedures (we have made them explicit). In each case, we spell out the invariant difference schemes and choose one that has exactly the same general solution as the ODE.

2. Lie point symmetries and first-order difference schemes

The point of view that we will be taking here is the same as in previous publications, e.g. [2–12] and references therein. Namely, Lie point symmetries of difference equations will be continuous point transformations $(x, y) \to (\tilde{x}, \tilde{y})$, taking solutions of the system (1.1) into solutions of the same sysetm. They will be induced by a Lie algebra of the vector fields of the same form (1.7) as for differential equations. The prolongation of the vector field will be different. No derivatives figure in equation (1.1); instead we prolong to other points on the lattice. In the case of the system (1.1), we have

$$\mathrm{pr}^D X = \xi(x, y)\partial_x + \phi(x, y)\partial_y + \xi(x_+, y_+)\partial_{x_+} + \phi(x_+, y_+)\partial_{y_+}. \tag{2.1}$$

The continuous limit (1.9) is recovered by putting

$$x_+ = x + h \qquad y_+ \equiv y(x_+) = y(x) + hy'(x) + \cdots \tag{2.2}$$

then, expanding into Taylor series

$$\begin{aligned} \xi(x_+, y_+) &= \xi(x, y) + h\xi_x(x, y) + hy'\xi_y(x, y) + \cdots \\ \phi(x_+, y_+) &= \phi(x, y) + h\phi_x(x, y) + hy'\phi_y(x, y) + \cdots \end{aligned} \tag{2.3}$$

and acting with $\mathrm{pr}^D X$ on a function of x, y, h and y_x (see equation (1.5)). We obtain

$$\mathrm{pr}^D X(F(x, y, h, y_x)) = \big\{\xi(x, y)\partial_x + \phi(x, y)\partial_y + [\phi_x + (\phi_y - \xi_x)y_x - \xi_y(y_x)^2]\partial_{y_x}\big\}F(x, y, h, y_x) + O(h). \tag{2.4}$$

Thus, we have

$$\lim_{h\to 0} \mathrm{pr}^D X = \mathrm{pr}\, X \tag{2.5}$$

as required.

Using the prescription

$$\mathrm{pr}^D X(E_a)|_{E_1=E_2=0} = 0 \tag{2.6}$$

we can find the symmetries of a given system (1.1). Instead, we shall start from an ODE and its known symmetries and construct the invariant difference scheme from the known vector fields.

For a one-dimensional symmetry algebra, we find invariants by solving the partial differential equation:

$$\big[\xi(x, y)\partial_x + \phi(x, y)\partial_y + \xi(x_+, y_+)\partial_{x_+} + \phi(x_+, y_+)\partial_{y_+}\big]F(x, y, x_+, y_+) = 0 \tag{2.7}$$

by the method of characteristics. The elementary invariants are

$$I_1^D = I_1^D(x, y) \qquad I_2^D = I_2^D(x_+, y_+) \qquad I_3^D = I_3^D(x, y, h, y_x) \qquad \frac{\partial I_3}{\partial y_x} \neq 0 \tag{2.8}$$

with h and y_x as in equation (1.5). Any two relations between expressions (2.8) will give an invariant difference scheme, for instance

$$I_3^D = F\big(I_1^D\big) \qquad I_1^D = I_2^D \tag{2.9}$$

with F chosen to obtain the correct continuous limit (we will drop the superscript D below).

If we have a two-dimensional symmetry algebra, we will obtain two invariants. If one of them is of the type I_3 in (2.8), we again obtain a (strongly) invariant difference scheme. If not, we must look for an invariant manifold, given in this case by the rank condition

$$\mathrm{rank}\begin{pmatrix}\xi_1(x, y) & \phi_1(x, y) & \xi_1(x_+, y_+) & \phi_1(x_+, y_+)\\ \xi_2(x, y) & \phi_2(x, y) & \xi_2(x_+, y_+) & \phi_2(x_+, y_+)\end{pmatrix} = 1. \tag{2.10}$$

We will see below for specific examples that for each invariant ODE we obtain invariant difference schemes with the same invariance group and the same general solution. Indeed, our choice of the invariant difference schemes

$$E_a(I_1, I_2, I_3) = 0 \qquad a = 1, 2 \tag{2.11}$$

will be guided by two considerations:

(i) To obtain the original ODE in the continuous limit.
(ii) To obtain a difference scheme that has exactly the same general solution as the original ODE (for any value of the lattice spacing $h = x_+ - x$).

3. Linear equations

Let us consider the first-order linear inhomogeneous ODE

$$y' = a(x)y + b(x). \tag{3.1}$$

For convenience, we redefine the given functions $a(x)$ and $b(x)$, putting $a(x) \equiv A'(x)$, $b(x) \equiv B'(x)\,\mathrm{e}^{A(x)}$. Equation (3.1) and its general solution are then written as

$$y' = A'(x)y + B'(x)\,\mathrm{e}^{A(x)} \tag{3.2}$$

$$y(x) = (B(x) + k)\,\mathrm{e}^{A(x)} \tag{3.3}$$

where k is the integration constant and the primes indicate x-derivatives.

Equation (3.2) has a two-dimensional Lie point symmetry group, generated by the vector fields

$$X_1 = e^{A(x)}\partial_y \qquad X_2 = [y - B(x)\,e^{A(x)}]\partial_y. \tag{3.4}$$

Now let us look for a difference scheme invariant under the group generated by the Lie algebra (3.4). The prolongations of X_1 and X_2 to the space $\{x, y, x_+, y_+\}$ have only two invariants x and x_+. However, they do allow an invariant manifold, given by the condition (2.10), which in this case reduces to

$$y_+\,e^{-A(x_+)} - y\,e^{-A(x)} - B(x_+) + B(x) = 0. \tag{3.5}$$

Adding an invariant lattice equation, e.g.,

$$x_+ - x = h \tag{3.6}$$

we obtain an invariant difference scheme (3.5), (3.6). Not only does this system reduce to the ODE (3.2) in the continuous limit, but equation (3.3) gives the exact general solution of the discrete system (3.5), (3.6) (for any value of h).

Note that while equation (3.5) is linear in y, it is not the difference equation one would get by the usual numerical discretization. To compare the two, let us reintroduce the discrete variable n, putting $x = x_n$, $x_+ = x_{n+1}$, $y = y_n$, $y_+ = y_{n+1}$.

A 'naive' discretization would be

$$\frac{y_{n+1} - y_n}{x_{n+1} - x_n} = A'(x_n)y_n + B'(x_n)\,e^{A(x_n)} \tag{3.7}$$

or possibly

$$\frac{y_{n+1} - y_n}{x_{n+1} - x_n} = \frac{A(x_{n+1}) - A(x_n)}{x_{n+1} - x_n}y_n + \frac{B(x_{n+1}) - B(x_n)}{x_{n+1} - x_n}\,e^{A(x_n)} \tag{3.8}$$

with $x_n = nh$ in both cases. The system (3.5), (3.6), on the other hand, is rewritten as

$$\begin{aligned} &y_{n+1}\,e^{-A(x_{n+1})} - y_n\,e^{-A(x_n)} - B(x_{n+1}) + B(x_n) = 0 \\ &x_{n+1} - x_n = h. \end{aligned} \tag{3.9}$$

All of the above discretizations coincide in the limit $h \to 0$; however, only the discretization (3.5), (3.6), i.e. (3.9), is exact in the sense that it has exactly the same general solution (3.3) as the ODE (3.1).

Similar comments hold for all the discretizations that we present below in sections 4–9. We shall not repeat them each time.

4. Separable equations

Let us consider the separable ODE

$$y' = f(x)g(y) \tag{4.1}$$

and for convenience redefine $f(x) \equiv A'(x)$, $g(y) \equiv 1/\dot{B}(y)$ (the prime is an x-derivative, the dot a y-derivative). The (implicit) general solution of equation (4.1) is

$$B(y) = A(x) + k \tag{4.2}$$

where k is a constant. The equation itself is rewritten as

$$y' = \frac{A'(x)}{\dot{B}(y)}. \tag{4.3}$$

Equation (4.3) has a two-dimensional symmetry group, generated by

$$X_1 = \frac{1}{\dot{B}(y)}\partial_y \qquad X_2 = \frac{1}{A'(x)}\partial_x. \tag{4.4}$$

The group invariants of the group generated by the algebra (4.4) in the discrete space are

$$I_1 = A(x_+) - A(x) \qquad I_2 = B(y_+) - B(y). \tag{4.5}$$

Using I_1 and I_2, we write an invariant difference scheme as

$$B(y_+) - B(y) - A(x_+) + A(x) = 0 \tag{4.6}$$

$$A(x_+) - A(x) = \epsilon(h) \tag{4.7}$$

where ϵ is some constant, satisfying $\epsilon(h) \to 0$ for $h \to 0$. Equation (4.2) clearly provides the general solution of equation (4.6).

As a specific example, let us choose

$$B(y) = y^M \qquad A(x) = x^N.$$

The difference scheme

$$y_+^M - y^M - x_+^N + x^N = 0 \tag{4.8}$$

$$x_+^N - x^N = \epsilon \tag{4.9}$$

is solved by

$$y = (x^N + k)^{1/M} \qquad x_n = \left(n\epsilon + x_0^N\right)^{1/N} \tag{4.10}$$

where we choose $\alpha > 0$, $x_0 \geqslant 0$. Clearly $y(x)$ as in (4.10) also solves the ODE obtained in the continuous limit, namely,

$$y' = \frac{N x^{N-1}}{M y^{M-1}}. \tag{4.11}$$

5. Exact equations

We will consider in this section exact equations, that is, equations of the form

$$y'(x) = -\frac{A(x, y)}{B(x, y)} \qquad \text{i.e.} \quad A(x, y)\,\mathrm{d}x + B(x, y)\,\mathrm{d}y = 0 \tag{5.1}$$

satisfying

$$A_y = B_x \qquad \text{i.e.} \quad A(x, y) = V_x(x, y) \quad B(x, y) = V_y(x, y) \tag{5.2}$$

for some function $V(x, y)$.

Equation (5.1) is invariant under a one-dimensional group generated by

$$X = B(x, y)\partial_x - A(x, y)\partial_y. \tag{5.3}$$

The general solution of (5.1) is given implicitly by the relation

$$V(x, y) = k \tag{5.4}$$

where k is an integration constant.

In the discrete case, equation (2.7) leads to the characteristic system

$$\frac{\mathrm{d}x}{V_y(x, y)} = -\frac{\mathrm{d}y}{V_x(x, y)} = \frac{\mathrm{d}x_+}{V_{y_+}(x_+, y_+)} = -\frac{\mathrm{d}y_+}{V_{x_+}(x_+, y_+)} \tag{5.5}$$

and hence to three invariants

$$I_1 = V(x, y) \qquad I_2 = V(x_+, y_+) \tag{5.6}$$

$$I_3 = \int \frac{\mathrm{d}x_+}{V_{y_+}(x_+, y_+(x_+, I_2))} - \int \frac{\mathrm{d}x}{V_y(x, y(x, I_1))}. \tag{5.7}$$

To obtain the integrals involved in I_3, we have solved equations (5.6) for y and y_+. The discrete version of equation (5.1) is

$$V(x_+, y_+) - V(x, y) = 0 \tag{5.8}$$

with (5.4) as its solution. Equation (5.7) can be used to define the invariant lattice.

As a specific example, consider the ODE

$$y' = \frac{1}{2(x+y)} - 1. \tag{5.9}$$

It is exact and we have

$$V = (x+y)^2 - x. \tag{5.10}$$

The vector field (5.3) in this case is

$$X = 2(x+y)\partial_x - (2(x+y) - 1)\partial_y. \tag{5.11}$$

The invariants in the discrete case are

$$I_1 = (x+y)^2 - x \qquad I_2 = (x_+ + y_+)^2 - x_+ \qquad I_3 = x_+ + y_+ - x - y. \tag{5.12}$$

The invariant difference scheme can be written as

$$(x_+ + y_+)^2 - (x+y)^2 = x_+ - x \tag{5.13}$$

$$x_+ + y_+ - x - y = \epsilon. \tag{5.14}$$

Returning to the general case, we see that the discrete analogue of an exact ODE is equation (5.8). The invariant lattice can be given by equation (5.7). In the continuous limit, we have $I_3 \to 0$, i.e., $x_+ \to x$ and (5.8) goes to

$$V_x + V_y y_x = 0. \tag{5.15}$$

6. Homogeneous equations

Let us consider the first-order ODE

$$y' = x^{k-1} F\left(\frac{y}{x^k}\right) \tag{6.1}$$

where F is an arbitrary smooth function and k is a real constant. This is the most general first-order ODE invariant under the scaling group

$$\tilde{y} = \mathrm{e}^{\lambda k} y \qquad \tilde{x} = \mathrm{e}^{\lambda} x \tag{6.2}$$

generated by the vector field

$$X = x\partial_x + ky\partial_y. \tag{6.3}$$

For convenience, we replace the function $F(t)$ in equation (6.1) by $F(t) = 1/\dot{H}(t) + kt$, so equation (6.1) is rewritten as

$$y' = \frac{x^{k-1}}{\dot{H}\left(\frac{y}{x^k}\right)} + k\frac{y}{x}. \tag{6.4}$$

The change of variables $(x, y(x)) \to (t, z(t))$ with

$$t = \frac{y}{x^k} \qquad z = \log x \qquad x = \mathrm{e}^z \qquad y = t\,\mathrm{e}^{kz} \tag{6.5}$$

will straighten out the vector field X and transform equation (6.1) into

$$z_t = \dot{H}(t). \tag{6.6}$$

Solving equation (6.6) and returning to the original variables, we obtain the general solution of equation (6.1) in the form

$$y(x) = x^k H^{-1}(\log x - C) \tag{6.7}$$

where C is an integration constant and H^{-1} is the function inverse to $H(t)$.

Let us now find the invariant difference scheme corresponding to equation (6.1). Prolonging the vector field X of equation (6.3) as in equation (2.1), we find three elementary invariants

$$I_1 = \frac{y}{x^k} \qquad I_2 = \frac{y_+}{x_+^k} \qquad I_3 = \frac{x_+}{x}. \tag{6.8}$$

Using them we write an invariant difference scheme as

$$\log I_3 - H(I_2) + H(I_1) = 0 \tag{6.9}$$

$$I_3 - 1 - \epsilon = 0. \tag{6.10}$$

More explicitly, we have

$$\log x_+ - \log x - H\left(\frac{y_+}{x_+^k}\right) + H\left(\frac{y}{x^k}\right) = 0 \tag{6.11}$$

$$x_+ - x - \epsilon x = 0 \tag{6.12}$$

where ϵ is some constant. The $\epsilon \to 0$ limit of the difference scheme (6.11), (6.12) is the ODE (6.4), as required, and its general solution is (6.7), together with

$$x_n = (\epsilon + 1)^n x_0. \tag{6.13}$$

As in the previous examples, the invariant difference scheme has the same exact solution as the original ODE.

As a specific example, we take $k = 1$ and the ODE

$$y' = \frac{x^2 + y^2}{xy}. \tag{6.14}$$

The difference scheme in this case is

$$\left(\frac{y_+}{x_+}\right)^2 - \left(\frac{y}{x}\right)^2 = 2\log\frac{x_+}{x} \qquad \frac{x_+ - x}{x} = \epsilon. \tag{6.15}$$

The solution of both (6.14) and (6.15) is

$$y = x\sqrt{2\log x - C} \tag{6.16}$$

(together with (6.13) in the discrete case).

7. Rotationally invariant equations

Let us consider a rotation in the (x, y) space. It is generated by

$$X = y\partial_x - x\partial_y. \tag{7.1}$$

The most general first-order ODE invariant under these rotations is

$$y' = \frac{K(\rho)y - x}{y + K(\rho)x} \qquad \rho = \sqrt{x^2 + y^2}. \tag{7.2}$$

To solve equation (7.2), we straighten out the vector field (7.1) by going to polar coordinates $(x, y(x)) \rightarrow (\rho, \alpha(\rho))$

$$x = \rho\cos\alpha \qquad y = \rho\sin\alpha. \tag{7.3}$$

Equation (7.2) reduces to

$$\alpha_\rho = -\frac{1}{\rho K(\rho)} \tag{7.4}$$

and we obtain

$$\alpha(\rho) = -\int \frac{1}{\rho K(\rho)}\, \mathrm{d}\rho. \tag{7.5}$$

For simplicity, let us restrict to the special case $K(\rho) = K =$ constant. We then obtain the solution of (7.2) as a logarithmic spiral

$$K\alpha + \log\rho = \log\rho_0 \tag{7.6}$$

or, in the original variables

$$\frac{1}{2}\log(x^2 + y^2) + K\arctan\frac{y}{x} = \log\rho_0. \tag{7.7}$$

Now let us consider the discrete case. The prolonged vector field is

$$\mathrm{pr}\, X = y\partial_x - x\partial_y + y_+\partial_{x_+} - x_+\partial_{y_+}. \tag{7.8}$$

This means that the pairs (x, y) and (x_+, y_+) will transform like vectors undergoing a rotation in a Euclidean plane. We can form four invariants

$$I_1 = x^2 + y^2 \qquad I_2 = x_+^2 + y_+^2 \qquad I_3 = xy_+ - x_+y \qquad I_4 = xx_+ + yy_+ \tag{7.9}$$

with one relation between them, namely,

$$I_3^2 + I_4^2 = I_1 I_2. \tag{7.10}$$

As an invariant difference scheme, we write

$$E_1 = \log\frac{I_2}{I_1} + 2K\arctan\frac{I_3}{I_4} = 0 \tag{7.11}$$

$$E_2 = \tfrac{1}{2}(I_2 - I_1) + K I_3 = 0 \tag{7.12}$$

with K being a constant. Using equation (2.2), we can check that the continuous limit of both these expressions is the ODE (7.2) (with K constant). In general, we have

$$\left|\frac{\partial(E_1, E_2)}{\partial(x_+, y_+)}\right|_{h\neq 0} \neq 0 \qquad \lim_{h\to 0}\left|\frac{\partial(E_1, E_2)}{\partial(x_+, y_+)}\right| = 0. \tag{7.13}$$

Expression (7.7), the exact solution of the ODE (7.2), is also an exact solution of the system (7.11), (7.12).

The situation is more transparent in polar coordinates (7.3). For K = constant, the ODE is

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\rho} = -\frac{1}{K\rho}. \tag{7.14}$$

The prolongation of the vector field corresponding to rotational invariance in the discrete case is

$$\mathrm{pr}\, X = \partial_\alpha + \partial_{\alpha_+}. \tag{7.15}$$

The invariants are

$$I_1 = \alpha_+ - \alpha \qquad I_2 = \rho_+ \qquad I_3 = \rho. \tag{7.16}$$

An invariant difference scheme corresponding to the ODE (7.14) is

$$\alpha_+ - \alpha = -\frac{1}{K}(\log \rho_+ - \log \rho) \tag{7.17}$$

$$\rho_+ - \rho = \epsilon. \tag{7.18}$$

The exact solution of (7.14) and (7.17) is given by equation (7.6). The lattice determined by equation (7.18) is uniform:

$$\rho_n = \epsilon n + \rho_0. \tag{7.19}$$

8. Invariant difference schemes on uniform lattices

Let us consider the ODE

$$y' = -\frac{A_x}{A_y} + f'(x)\frac{1}{A_y} \qquad A_y \neq 0 \tag{8.1}$$

where $A(x, y)$ and $f(x)$ are some smooth functions. Equation (8.1) is invariant under transformations generated by

$$X = \frac{1}{A_y(x, y)}\partial_y. \tag{8.2}$$

As a matter of fact, any first-order ODE invariant under transformations generated by a vector field of the form

$$X = \phi(x, y)\partial_y \tag{8.3}$$

can be reduced to the form (8.1).

The general solution of equation (8.1) can be written implicitly as

$$A(x, y) = f(x) + C. \tag{8.4}$$

In the discrete case, the first prolongation of the vector field (8.2) has three invariants; they can be chosen to be

$$I_1 = x_+ \qquad I_2 = x \qquad I_3 = A(x_+, y_+) - A(x, y). \tag{8.5}$$

An invariant difference scheme on a uniform lattice, having the ODE as a continuous limit, is

$$A(x_+, y_+) - A(x, y) = f(x_+) - f(x) \tag{8.6}$$

$$x_+ - x = h. \tag{8.7}$$

Its solution is given by equation (8.4) together with

$$x_n = hn + x_0. \tag{8.8}$$

As a specific example, let us choose

$$X = xy\partial_y. \tag{8.9}$$

The invariant ODE in this case is

$$y' = \frac{y}{x}\log y + q'(x)xy. \tag{8.10}$$

The corresponding difference scheme is

$$\frac{1}{x_+}\log y_+ - \frac{1}{x}\log y - q(x_+) + q(x) = 0 \qquad x_+ - x = h. \tag{8.11}$$

The solution of both (8.10) and (8.11) is

$$y = \mathrm{e}^{x(C+q(x))} \tag{8.12}$$

where C is an integration constant.

The vector field (8.2) is not the only one compatible with a uniform lattice. Another one is

$$X = \partial_x + \phi(x, y)\partial_y \tag{8.13}$$

for any function $\phi(x, y)$. As an example, let us take

$$X = \partial_x + x^a y^b \partial_y. \tag{8.14}$$

The corresponding invariant first-order ODE is

$$y' = (k(\zeta) + x^a)y^b \qquad \zeta = \frac{1}{a+1}x^{a+1} + \frac{1}{b-1}y^{-b+1} \tag{8.15}$$

where k is any function of ζ. For $k = k_0 =$ constant. The solution of equation (8.15) is

$$y = (1-b)^{\frac{1}{1-b}}\left[k_0 x + \frac{1}{a+1}x^{a-1} + C\right]^{\frac{1}{1-b}} \tag{8.16}$$

where C is an integration constant.

The invariants of the discrete prolongation of the vector field (8.14) are

$$I_1 = x_+ - x \qquad I_2 = \frac{x^{a+1}}{a+1} + \frac{y^{-b+1}}{b-1} \qquad I_3 = \frac{x_+^{a+1}}{a+1} + \frac{y_+^{-b+1}}{b-1}. \tag{8.17}$$

We write an invariant difference scheme as

$$I_2 - I_3 - k_0 I_1 = 0 \qquad I_1 = h \tag{8.18}$$

with k_0 and h constant. More explicitly (8.18) is

$$\frac{1}{1-b}\left[y_+^{1-b} - y^{1-b}\right] - \frac{1}{1+a}\left[x_+^{a+1} - x^{a+1}\right] - k_0(x_+ - x) = 0 \tag{8.19}$$

$$x_+ - x = h. \tag{8.20}$$

The continuous limit of (8.19) is the ODE (8.15). Moreover, equation (8.16) is the exact solution of (8.19) for any value of h (not just the limit $h \to 0$).

9. Invariant difference schemes on exponential lattices

The vector field (8.2) is also compatible with an invariant scheme of the form (8.6) on the exponential lattice

$$\frac{x_+ - x}{x} = \epsilon \qquad \text{i.e.} \quad x_n = (\epsilon + 1)^n x_0 \tag{9.1}$$

with $\epsilon =$ constant. Taking the limit $\epsilon \to 0$ we again obtain the ODE (8.1).

Another symmetry, leaving the lattice (9.1) invariant, is generated by the vector field

$$X = x\partial_x + \phi(x, y)\partial_y \tag{9.2}$$

with $\phi(x, y)$ arbitrary.

As an example, consider

$$X = x\partial_x + x^a y^b \partial_y \qquad a \neq 0 \quad b \neq 1. \tag{9.3}$$

The corresponding invariant ODE is

$$y' = k(\zeta)\frac{1}{x} y^b + x^{a-1} y^b \qquad \zeta = \frac{1}{a} x^a + \frac{1}{b-1} y^{1-b}. \tag{9.4}$$

For $k = k_0$ constant, the solution is

$$y = (1 - b)^{\frac{1}{1-b}} \left[k_0 \log x + \frac{x^a}{a} + C \right]^{\frac{1}{1-b}}. \tag{9.5}$$

In the discrete case, the prolongation of X has three invariants

$$I_1 = \frac{x_+}{x} \qquad I_2 = \frac{y^{1-b}}{b-1} + \frac{x^a}{a} \qquad I_3 = \frac{y_+^{1-b}}{b-1} + \frac{x_+^a}{a}. \tag{9.6}$$

An invariant scheme having (9.4) with $k = k_0$ as its limit and (9.5) as its solution is

$$I_2 - I_3 - k_0 \log I_1 = 0 \qquad I_1 - 1 = \epsilon \tag{9.7}$$

i.e.,

$$\frac{y_+^{1-b}}{1-b} - \frac{y^{1-b}}{1-b} - \frac{x_+^a}{a} + \frac{x^a}{a} - k_0(\log x_+ - \log x) = 0 \tag{9.8}$$

$$x_+ - x = \epsilon x_+. \tag{9.9}$$

10. Conclusions

Looking directly for symmetries of a difference scheme of the type (1.1) is not a particularly fruitful enterprise. As in the case of first-order ODEs, one gets an underdetermined system of equations. Infinitely many solutions exist, but there is no algorithm for finding them.

We have taken the complementary point of view. We have postulated the form of a vector field, then found ODEs and difference schemes, invariant under the corresponding symmetry group. The symmetry makes it possible to solve the ODE exactly analytically. The invariant differential scheme then has the same solution. More precisely, the symmetry leads to a family of difference schemes, one of which has solutions coinciding with those of its continuous limit (the original ODE).

Essentially, we have constructed a partial 'catalogue' of exactly solvable two-point schemes. This corresponds to a list of exactly solvable first-order ODEs: linear equations, separable equations, exact equations, homogeneous equations, etc. The 'complete list' is infinite: for any chosen realization of a one- or two-dimensional Lie algebra, we can construct an invariant ODE and an invariant difference scheme.

Without the symmetries to guide us, the obtained difference schemes are not obvious at all. Starting from an ODE (1.6) and discretizing in a standard way, i.e., replacing the derivative $\mathrm{d}y/\mathrm{d}x$ by a discrete derivative and writing

$$\frac{y_{n+1} - y_n}{x_{n+1} - x_n} = F(x_n, y_n) \qquad x_n = hn + x_0 \tag{10.1}$$

we would lose virtually all symmetries. Moreover, the exact solution of equation (10.1) would differ from that of the ODE (1.6) by terms of the order h. For our symmetry dictated difference schemes, the exact solutions coincide with those of the ODEs. The main result of this paper can be summed up as follows.

Theorem. *For every first-order ODE there exists an invariant two-point difference scheme with exactly the same general solution as the ODE.*

Proof. Consider the ODE (1.6) and assume that we know its general solution in the form of a first integral

$$h(x, y) = K \qquad h_y \neq 0. \tag{10.2}$$

The general element (1.7) of the symmetry algebra of equation (1.6) must annihilate the function $h(x, y)$. The equation $Xh = 0$ implies that X has the form

$$X = \xi(x, y)\left(\partial_x - \frac{h_x}{h_y}\partial_y\right) \tag{10.3}$$

where $\xi(x, y)$ is an arbitrary smooth function.

Let us now find the invariants of the group action induced by the vector field (10.3) in the space (x, x_+, y, y_+). They are obtained by solving the characteristic system:

$$\frac{\mathrm{d}x}{\xi(x, y)} = \frac{\mathrm{d}x_+}{\xi(x_+, y_+)} = \frac{h_y(x, y)\,\mathrm{d}y}{-\xi(x, y)h_x(x, y)} = \frac{h_{y_+}(x_+, y_+)\,\mathrm{d}y_+}{-\xi(x_+, y_+)h_{x_+}(x_+, y_+)}. \tag{10.4}$$

The invariants hence are

$$I_1 = h(x, y) \qquad I_2 = h(x_+, y_+)$$
$$I_3 = \int \frac{\mathrm{d}x}{\xi(x, y(x, I_1))} - \int \frac{\mathrm{d}x_+}{\xi(x_+, y_+(x_+, I_2))} \tag{10.5}$$

where the function $\xi(x, y)$ can be freely chosen.

A difference scheme that has (10.2) as its general solution is

$$I_1 = I_2 \qquad I_3 = c \qquad (c = \text{constant}). \tag{10.6}$$

QED

Three comments are in order here.

1. The proof given above is not a constructive one. It assumes that we already know the general solution (10.2) of the ODE. This was not assumed in the rest of this paper, where we constructed the difference schemes using only one, or sometimes two elements of the symmetry algebra of an ODE.
2. The arbitrariness in the function $\xi(x, y)$ can be put to good use in the choice of lattices. For instance, choosing $\xi(x, y) = 1$, we obtain a uniform lattice as in (8.8). Choosing $\xi = x$, we obtain an exponential lattice as in equation (9.1).
3. The theorem and the results of this paper are specific to first-order ODEs and two-point difference schemes.

Let us compare with the case of second-order ODEs and three-point difference schemes. S Lie classified all (complex) second-order ODEs into equivalence classes according to their Lie point symmetries [13]. A similar classification of three-point difference schemes is much more recent [11]. It was shown [12] that if a three-point difference scheme has a symmetry group of dimension 3 (or larger) with at least a two-dimensional subalgebra of Lagrangian symmetries, then the scheme can be integrated analytically. A crucial element in the integration was the existence of a Lagrangian and the interpretation of the difference scheme as a discrete analogue of an Euler–Lagrange equation [11, 12].

The fact that we obtain differential equations and difference schemes that have identical symmetries and solutions has interesting implications. It suggests a certain duality between continuous and discrete descriptions of physical phenomena. Thus, exactly the same physical predictions may be described by some continuous curve, or by a series of points on this curve, distributed with an arbitrary density.

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